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The nonautonomous function-theoretic center problem

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Abstract. We study the linearizability and stability of a nonautonomous dynamical system in the neighborhood of a neutral fixed point. Our results generalize the classical results of Schröder and Siegel in the case when the linear part of the mapping is an irrational rotation, well known results in the rational case and the fundamental result on the representation of the system as a translation in the neighborhood of a fixed point at infinity.

Keywords: Center problem, nonautonomous dynamical system, linearizability, rotation, conformal mapping.

1. Introduction: a description of the main results and conditions

In this article we study the behavior of nonautonomous dynamical systems (NDS) which converge to a limit in the neighborhood of a fixed point. Such NDS's arise in many problems in the theory of differential equations, mechanics, mathematical and statistical physics (see for example [**6**]-[**8**]). They also arise in the theory of the Riemann ζ-function [**9**], where with the help of an appropriate two dimensional NDS it was shown that the ζ-function can be represented in the critical strip as a product of second order matrices, this is an analogous of the Euler product to the right of the critical strip [**9**]. Furthermore in [**9**] the correctness of the Riemann hypothesis on the zeros of the ζ-function is transformed to the study of the stability of the fixed point of this NDS.

We consider a NDS with discrete time, which is defined by a sequence of conformal mappings $F^{(1)}, F^{(2)}, \dots$,

$$F^{(n)}: z \to z' = F^{(n)}(z) = \lambda z + f^{(n)}(z), \tag{1.1}$$

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where

$$n = 1, 2, \dots ; f^{(n)}(z) = \sum_{k=2}^{\infty} f_k^{(n)} z^k,$$

and $f_k^{(n)}$ does not depend on z. The mappings $F^{(n)}$ are defined in a fixed neighborhood of $0 \in \mathbb{C}$. The NDS is defined by first applying $F^{(1)}$, then applying $F^{(2)}$ to it's image, and so on. We assume that the NDS is convergent, that is that as $n \to \infty$ the sequence of mappings $F^{(n)}$ in equation (1.1) smoothly approach a limit mapping

$$F: z \to z' = F(z) = \lambda z + f(z),$$
$$f(z) = \sum_{k=2}^{\infty} f_k z^k .$$

Furthermore, we assume f_k does not depend on z,

$$f(z) = \lim_{n \to \infty} f^{(n)}(z), \tag{1.2}$$

and the convergence in equation (1.2) is uniform for all z in some fixed neighborhood of the fixed point $z = 0$. Throughout the article we will assume that the fixed point $z = 0$ is neutral, i.e. $|\lambda| = 1$. The main problem which is studied in this article is the generalization to the nonautonomous case of the classical center problem of classifying the behavior of the iterates of a single mapping near a neutral fixed point. In the classical problem (first studied by Schröder [**10**]) the behavior around a fixed point is studied for the iteration of one mapping. In the nonautonomous case, this problem was first studied in [**4**]-[**6**].

We explain in what sense we want to study the center problem (CP) for a NDS. We need to find a sequence of changes of variables

$$\zeta^{(n)} = P^{(n)}(z) = z + p^{(n)}(z); \; p^{(n)}(z) = \sum_{k=2}^{\infty} p_k^{(n)} z^k; \; n = 1, 2, \dots \tag{1.3}$$

($p_k^{(n)}$ does not depend on z) which are analytic and invertible in a neighborhood (not depending on n) $|z| \leq r_1 (r_1 > 0)$. Furthermore, the changes of variables converge (uniformly) in the neighborhood to the change of variables

$$\zeta = P(z) = z + p(z);\ p(z) = \sum_{k=2}^{\infty} p_k z^k \tag{1.4}$$

such that there exists a constant $r_2 > 0$ such that for any $n \in \mathbb{N}$ the mapping

$$D^{(n)} := P^{(n+1)} \circ F^{(n)} \circ (P^{(n)})^{-1} \tag{1.5}$$

is defined in the disc $\left|\zeta^{(n)}\right| \le r_2$ and has the form

$$D^{(n)} : \zeta^{(n)} \to \zeta^{(n+1)} = \lambda \zeta^{(n)}\ . \tag{1.6}$$

If such a sequence of change of variables exists, then we say that the NDS defined by equation (1.1) is *conjugate to a rotation or linearizable*, otherwise we say that the NDS is *not conjugate to a rotation or not linearizable.* In the special case when all the $F^{(n)}$'s coincide with one another then we arrive at the classical CP. The statement and the first result about CP for a NDS are due to Pustyl'nikov [**4**]-[**5**], who proved a generalization of Siegel's theorem [**11**]-[**12**]. Namely, he showed that under certain conditions a NDS is conjugate to a rotation if the Siegel condition holds for the parameter λ:

$$|\lambda| = 1, \left|\lambda^q - 1\right|^{-1} \le c_0 q^2 \text{ for } q = 1, 2, \dots\ . \tag{1.7}$$

The proof of [**5**]-[**6**] assumes that condition (L) formulated below, is fulfilled. Condition (L) characterizes the smoothness of the convergence of the $F^{(n)}$ as $n \to \infty$. We note that the rate of convergence can be arbitrarily slow. A generalization of theorem 1 from [**5**] is formulated and proved in [**6**] (part 2, chapter V, §2) to the case when the parameter $\lambda = \lambda^{(n)}$ in equation (1.1) depends on n and the mappings $F^{(n)}$ have the form

$$\begin{aligned} &F^{(n)} : z \to z' = F^{(n)}(z) = \lambda^{(n)} z + f^{(n)}(z), \text{ where} \\ &n = 1, 2, \dots\ , \text{ and } f^{(n)}(z) = \sum_{k=2}^{\infty} f_k^{(n)} z^k \end{aligned} \tag{1.8}$$

where the $\lambda^{(n)} \to \lambda$ as $n \to \infty$, λ satisfies equation (1.7) and the infinite product $\prod_{n=1}^{\infty} \lambda^{(n)}/\lambda$ converges absolutely. In this case it is clear, that instead of requiring the change of variables given by equation (1.3), we need the more general form

$$\zeta^{(n)} = P^{(n)}(z) = \mu_n z + p^{(n)}(z);\; p^{(n)}(z) = \sum_{k=2}^{\infty} p_k^{(n)} z^k;\; n = 1, 2, \ldots \quad (1.9)$$

where $\mu_n \to 1$.

We remark that the set of λ in the unit circle which satisfy condition (1.7) form a set of full measure. In this article which we prove several results about the CP for a NDS analogous to a classical result about the CP.

For the first problem we consider we assume that λ is not a root of unity. In this case we construct the sequence (1.3) of changes of variables in the form of formal power series in z so that the equalities (1.4, 1.6) hold in the sense of equalities of formal power series.

The second problem we consider is when $\lambda = \exp 2\pi i\Delta$ and Δ is a Siegel irrational number. We prove that a NDS is conjugate to a rotation in the case when Δ is an irrational number such that for any $m \in \mathbb{Z}$ and any $k \in \mathbb{N}$ the following holds:

$$|k\Delta - m|^{-1} \leq C_0 k^{\mu} \quad (1.10)$$

where C_0 and μ are positive constants not depending on m or k.

The next problems concern the case when λ is a root of unity. In this case we prove that a NDS is unstable in the sense of Lyapunov and not conjugate to a rotation.

Finally we study the system (1.1) in inverse coordinates in a neighborhood of the fixed point in the case λ is a root of unity.

In the classical case the first problem corresponds to [**10**] (Schröder series), the second problem corresponds to the theorems of Siegel [**11**]-[**12**] and Bruno [**1**], the third corresponds to [**14**] and [**3**] and last problem was studied in [**2**]. Nice surveys of the classical CP can be found in [**2**] and [**13**]. Condition (1.10) under which conjugacy to a rotation is proven is more general than condition (1.7) and coincides with it in the case $\mu = 2$. The set of $\lambda \in \mathbf{S}^1$ for which we have shown that a NDS is conjugate or not conjugate to a rotation is not all of $\mathbf{S}^1$. The question remains open for nonrational $\lambda \in \mathbf{S}^1$ which do not satisfy condition

(1.10). In the classical CP the conjugacy with a rotation is proven for all $\lambda \in \mathbf{S}^1$ which satisfy the Bruno condition [**1**], and if λ does not satisfy the Bruno condition then Yoccoz has shown that there exists a polynomial who's leading term is λz which is not conjugate to a rotation [**15**].

The classical Schröder equation

$$u(\lambda z) = \lambda u(z) + f(z), \tag{1.11}$$

plays an important role in [**10**]. It is well known that in the case λ is not a root of unity this equation has a formal solution $u(z)$, while in the case λ satisfies the Siegel condition, the solution is analytic. In sections 3 and 4 of this paper we consider the following generalization of Schröder's equation:

$$u(\lambda z, y+1) = \lambda u(z,y) + f(z,y)\ . \tag{1.12}$$

Here $u(z,y)$ is the unknown function and $f(z,y)$ is a fixed function which is analytic in z in a given neighborhood of the point $z=0$ and satisfies $f(0,y) = (\partial f/\partial z)(0,y) = 0$. Several additional assumptions are made. We assume that the functions $f(z,y)$ and $u(z,y)$, considered as functions of y, are defined on a domain $Y \subset \mathbb{R}$ which includes all the points $y+n, n = 1,2,\ldots$ whenever $y \in Y$. Furthermore, we assume that the function $f(z,y)$ converges smoothly enough to a limit function $f(z,\infty)$. This notion will be made precise later. We want to find a solution $u(z,y)$ of equation (1.12) which is defined for all $y \in Y$, which is analytic in z in a neighborhood of $z=0$ (not depending on y). Furthermore, for any z from this neighborhood, the function $u(z,y)$ must converge smoothly enough as $y \to \infty$ to an analytic function $u(z,\infty)$ such that we can estimate the modulus $|u(z,y)|$ of this solution and its differences in y through the moduli and difference of the functions $f(z,y)$ (theorems 3.3, 3.4). If we reduce to the special case, when the functions $f(z,y) = f(z), u(z,y) = u(z)$ do not depend on y, then equation (1.12) reduces to Schröder's equation.

Equation (1.12) was introduced in [**5**] (section 2, lemmas 2.1 and 2.3) where, in the case that λ satisfies condition (1.7) a solution $u(z,y)$

was constructed satisfying the requirements described in the last paragraph. In this article (§3) the solution from [5] is used to study of the linearizability of NDS's for all values of λ except when λ is a root of unity (§4).

Another important idea which is used in this article for the proof of the linearizability of a NDS is the correct description of the smooth convergence of the NDS (1.1) as $n \to \infty$. For the proof of the formal conjugacy to a rotation (§3, theorem 4.1) it is enough only to require the convergence of the arbitrary first differences of the functions $f^{(n)}(z)$:

$$\sum_{n=1}^{\infty} \sup_{|z|\leq r_0} \left| f^{(n+1)}(z) - f^{(n)}(z) \right| < \infty \, . \tag{1.13}$$

However, for the proof of the analytic conjugacy to a rotation condition (1.13) is not enough. We additionally to require some estimates on the convergence of the absolute values of the higher differences of the function $f^{(n)}(z)$. These estimates are quite restrictive, it is impossible to effectively state them and they are not applicable to some natural NDS of more general form, for example:

$$F^{(n)}: z \longrightarrow z' = F^{(n)}(z) = \lambda z + g(z) + \frac{h(z)}{n^{\alpha}} \tag{1.14}$$

where α is an arbitrary positive constant and $g(0) = dg/dz(0) = h(0) = dh/dz(0) = 0$. In [5], to overcome this problem the condition (L) was introduced. Condition (L) consists of the following: for $|z| \leq r_0, y \geq 1, (r_0 > 0)$ there exists a function $F(z,y) = \lambda z + f(z,y)$ such that

(i) $f(z,n) = f^{(n)}(z)$;

(ii) in the disk $|z| \leq r_0$ there exists an analytic function

$$f(z,\infty) = f^{(\infty)}(z) = \lim_{y\to\infty} f(z,y);$$

(iii) the power series for $f(z,y)$ in z begins with second order terms;

(iv) for a certain $\beta > 0$ the function $f(z, x^{-\beta})$ has l continuous partial derivatives in x for $0 \leq x \leq 1$, (at the endpoints $x = 0$ and $x = 1$ we assume that one sided partial derivatives exist).

Part (iv) of condition (L) characterized the level of smoothness of the convergence of $F^{(n)}$ to their limit as $n \to \infty$. The main condition of part (iv) is the existence of l partial derivatives from the right at the point $x = 0$: one can always embed the system of mappings $F^{(n)}$ depending on a discrete parameter as smoothly as one wants in a system of mappings depending of a smooth parameter such that the function $f(z, x^{-\beta})$ will have the needed number of partial derivatives for all $x \neq 0$.

Condition (L) does not affect the speed of convergence of the $F^{(n)}$: this speed can be arbitrarily slow if β in part (iv) is sufficiently large. Condition (L) holds for the system (1.14): in this case $\beta = 1/\alpha$ and

$$f(z,y) = g(z) + \frac{h(z)}{y^{\alpha}} = f(z, x^{-\beta}) = g(z) + xh(z) \tag{1.15}$$

and condition (L) holds.

For a sequence of mappings $F^{(n)}$ given by equation (1.8) the following additional condition (in addition to (L)) is required [**6**] (part 2, chapter V, §2): there exists a function $\lambda(y)$ such that

(v) $\lambda(n) = \lambda^{(n)}, \lim_{y\to\infty} \lambda(y) = \lambda$;

(vi) the function $\Lambda(y) = \prod_{k=0}^{\infty} \lambda(y+k)/\lambda$ is defined for $y \geq 1$, the product is absolutely convergent and $\lim_{y\to\infty} \Lambda(y) = 1$;

(vii) for some $\beta > 0$ the function $\Lambda(x^{-\beta})$ has l continuous derivatives in x for $0 \leq x \leq 1$.

In the theorems on the analytic conjugacy to a circle rotation and on the stability of the point $z = 0$ (§4), we require that condition (L) is fulfilled for $l = 16$. In the proof of the nonconjugability to a circle rotation and the instability of the point $z = 0$ of the NDS's (1.1) or (1.8) in the case that λ is a root of unity (§4) we do not require condition (L), but only the converges of $F^{(n)}$ to the limit mapping F.

In section 6 we generalize to the nonautonomous case the important theorem on the iteration of a fixed mapping in the neighborhood of an infinite fixed point ($w = \infty$) and its application to the study of a conformal mapping, whose linear part is a rational rotation, near the point $z = 0$ [**2**]. In theorem 6.1 we consider a sequence of conformal mappings $G^{(n)}$, in the domain $Re(w) \geq \kappa_0$ having the form

$$G^{(n)}(w) = w + 1 + \frac{b_n}{w} + g^{(n)}(w), (n = 1, 2, \dots)\ ,$$

where b_n is a constant, and $g^{(n)}(w) = O(w^{-2})$. Our main result here is that there exists a change of variables $\Phi^{(n)}(w), (n = 1, 2, \dots)$ which bring the mappings $G^{(n)}$ to a translation by 1:

$$\Phi^{(n+1)} \circ G^{(n)}(w) = \Phi^{(n)}(w) + 1 \ .$$

In particular when all the $G^{(n)}(w)$ are all identical, then the $\Phi^{(n)}(w)$ are all identical and our theorem reduces to the one proven in [**2**]. Finally we mention that the study of a sequence of mappings $G^{(n)}(w)$ in a neighborhood of the point $w = \infty$ is the main factor in proving the Lyapunov instability and nonconjugability of the systems (1.1) and (1.8) at the point $z = 0$ when λ is a root of unity (theorems 5.1, 5.3, 5.5 and corollaries 5.2, 5.4, 5.6).

2. Main definitions and terminology

1. Suppose that for the function $h(z, y)$, defined for $|z| \leq r_0$ and $y \geq 1$, the limit $\lim_{y\to\infty} h(z, y) = h(z, \infty)$ exists and if the function $h(z, x^{-\beta})$ has $s \geq 0$ derivatives with respect to x for $0 \leq x \leq 1$ then

$$|h(z,y)|_s = \left|h(z,x^{-\beta})\right|_s = \sup_{0\leq\sigma\leq s}\ \sup_{0\leq x\leq 1}\ \sup_{|z|\leq r_0} \left|\left(\frac{\partial}{\partial x}\right)^{\sigma} h(z,x^{-\beta})\right| ,$$

where

$$\left(\frac{\partial}{\partial x}\right)^{0} h(z,x^{-\beta}) = h(z,x^{-\beta}) \ .$$

2. Let Y be an arbitrary set of reals satisfying $y + n \in Y$ for all $y \in Y$ and all $n \in \mathbb{Z}^+$ (the non negative reals). The two main examples which are used in this work are $Y = \mathbb{N}$ the natural numbers and $Y = \{y \geq 1\}$.

3. For any natural number s we recursively define the sth difference $\Delta^s h(z, y)$ of the function $h(z, y)$, defined in the domain $|z| \leq r_0, y \in Y$, by:

$$\begin{aligned}
\Delta^1 h(z,y) &= h(z,y+1) - h(z,y)\\
\Delta^2 h(z,y) &= \Delta^1(z,y+1) - \Delta^1 h(z,y)\\
&\ \vdots\\
\Delta^s h(z,y) &= \Delta^{s-1}(z,y+1) - \Delta^{s-1} h(z,y) \ .
\end{aligned}$$

4. Suppose that the sequence of formal power series

$$g^{(n)}(z) = \sum_{k\geq 0} g_k^{(n)} z^k, (n = 1, 2, \dots)$$

and the power series $g^{(\infty)}(z) = \sum_{k\geq 0} g_k^{(\infty)} z^k$ are given and the coefficients $g_k^{(n)}$ do not depend on z for all $n \leq \infty$ and all k. We say the sequence $g^{(n)}(z)$ converges as $n \to \infty$ to the series $g^{(\infty)}(z) = \lim_{n\to\infty} g^{(n)}(z)$ if for each $k \geq 0$ we have $\lim_{n\to\infty} g_k^{(n)} = g_k^{(\infty)}$.

5. Suppose that $f(z, y) = \sum_{k=2}^{\infty} f_k(y) z^k$ is a formal power series in z with coefficients $f_k(y)$, not depending on z. We say that the formal power series $u(z, y) = \sum_{k=2}^{\infty} u_k(y) z^k$ is a formal solution of equation (1.12) if, when it is applied to equation (1.12) then for all $k \geq 2$ the coefficients of z^k coincide as functions of y for all $y \in Y$.

6. The fixed point $z = 0$ is call *stable in the sense of Lyapunov* for a NDS given by a sequence of mappings $F^{(n)}: n = 1, 2, \dots$, if for any $\epsilon > 0$ there is a $\delta > 0$ so that

$$\left| F^{(n)} \circ \cdots \circ F^{(1)} z \right| < \epsilon$$

for any n and any $|z| \leq \delta$.

7. The fixed point $z = 0$ is call *uniformly stable in the sense of Lyapunov* for a NDS given by a sequence of mappings $F^{(n)}: n = 1, 2, \dots$, if for any $\epsilon > 0$ there is a $\delta > 0$ and for all positive integers $n_0 < n_1$ so that

$$\left| F^{(n_1)} \circ \cdots \circ F^{(n_0)} z \right| < \epsilon$$

for any $|z| \leq \delta$.

3. The functional equation

We consider equation (1.12), in which $u(z, y)$ is a unknown function, $f(z, y) = \sum_{k=2}^{\infty} f_k(y) z^k$ is a given z-analytic function in the domain $|z| \leq r_0$, $y \in Y$ and $f_k(y)$ does not depend on z.

Theorem 3.1. *Suppose that $|\lambda| = 1$ and λ is not a root of unity. Suppose further that for each $y \in Y$ the limit* $\lim_{n\to\infty} f(z, y + n) = f_\infty(z, y)$ *exists* *and the following inequality holds:*

$$\sum_{n=0}^{\infty} \sup_{|z|\le r_0} \left|\Delta^1 f(z, y+n)\right| < \infty . \tag{3.1}$$

Then there exists a formal solution $u(z,y)$ of equation (1.12) given by the formal power series

$$u(z,y) = \sum_{k=2}^{\infty} u_k(y) z^k . \tag{3.2}$$

The coefficients $u_k(y)$ have the form

$$u_k(y) = \frac{f_k(y) + \sum\limits_{n=0}^{\infty} (f_k(y+n+1) - f_k(y+n))\lambda^{(k-1)(n+1)}}{\lambda^k - \lambda} \tag{3.3}$$

and for each $y \in Y$ the limit

$$\lim_{n\to\infty} u(z, y+n) = u_\infty(z,y) , \tag{3.4}$$

exists where $n \in \mathbb{N}$ and $u_\infty(z,y)$ is a formal power series representing the formal solution of the equation

$$u_\infty(\lambda z, y) = \lambda u_\infty(z,y) + f_\infty(z,y) . \tag{3.5}$$

Proof. We apply the Cauchy error estimate in the domain $|z| \le r_0$ to the z-analytic function $f(z, y+n+1) - f(z, y+n)$. This yields:

$$|f_k(y+n+1) - f_k(y+n)| \le \sup_{|z|\le r_0} \left| \frac{\Delta^1 f(z, y+n)}{r_0^k} \right| .$$

Thus, using equation (3.1), the series on the right hand side of equation (3.3) converges, $u_k(y)$ is properly defined and the following equality holds:

$$\lim_{n\to\infty} u_k(y+n) = \frac{f_{\infty,k}(y)}{\lambda^k - \lambda} . \tag{3.6}$$

Here $f_{\infty,k}(y)$ is the kth coefficient of the z-Taylor series expansion $f_\infty(z,y) = \sum_{k=2}^{\infty} f_{\infty,k}(y) z^k$ of the z-analytic function $f_\infty(z,y)$ in the domain $|z| \le r_0$. The coefficient of z^k in the formal power series $u_\infty(z,y)$ is $u_{\infty,k} = \lim_{n\to\infty} u_k(y+n)$, thus using equation (3.6) we see that $u_\infty(z,y)$ formally satisfies equation (3.5). Thus to prove theorem 3.1 we must show that the formal power series (3.2) with coefficients (3.3) formally satisfy equation (1.12). This fact was proven in [**5**] (Lemma 2.1, formula 2.6). □

Corollary 3.2. *For any $s \in \mathbb{N}$ the coefficient $u_k(y)$ of the formal solution (3.2) of equation (1.12) defined by equation (3.3) satisfies:*

$$\Delta^s u_k(y) = \frac{1}{\lambda_k - \lambda}(\Delta^s f_k(y) + \sum_{n=0}^{\infty} \lambda^{(k-1)(n+1)} \Delta^{s+1} f_k(y+n)) \ .$$

Theorem 3.3. *We suppose the assumptions of theorem 3.1. Additionally we suppose that $\lambda = e^{2\pi i \Delta}$, where Δ is a real number, satisfies condition (1.10) (with integer constant $\mu > 0$), θ is a positive constant satisfying $0 < \theta < 1$, and s is a natural number. Then equations (3.2) and (3.3) define the solution $u(z,y)$ of equation (1.12), which is z-analytic in the domain $|z| < r_0(1-\theta)$, $y \in Y$. In this domain the following inequalities hold:*

$$|u(z,y)| \leq \frac{C_0 \mu!}{4\theta^{\mu+1}} \sup_{|z| \leq r_0} \left(|f(z,y)| + \sum_{n=0}^{\infty} \left| \Delta^1 f(z, y+n) \right| \right), \tag{3.7}$$

$$|\Delta^s u(z,y)| \leq \frac{C_0 \mu!}{4\theta^{\mu+1}} \sup_{|z| \leq r_0} \left(|\Delta^s f(z,y)| + \sum_{n=0}^{\infty} \left| \Delta^{s+1} f(z, y+n) \right| \right), \tag{3.8}$$

where C_0 is a constant defined in equation (1.10). Furthermore equation (3.4) holds, and the z-analytic function $u_\infty(z,y)$ satisfies equation (3.5).

Proof. For any $k \in \mathbb{N}$, we choose $\hat{m} \in \mathbb{Z}$ such that $|k\Delta - \hat{m}| < 1/2$. Using $4|k\Delta - \hat{m}| \leq \left|\lambda^k - 1\right|$ and equation (1.10) we get

$$\left|\lambda^k - 1\right|^{-1} < \frac{C_0}{4} k^\mu ; k = 1, 2, \ldots \ . \tag{3.9}$$

Now applying the estimate (3.9) to equation (3.3) and using equation (3.2) and the Cauchy estimate in the domain $|z| \leq r_0(1-\theta)$ gives:

$$|u(z,y)| \leq$$

$$\leq \frac{C_0}{4} \sup_{|z| \leq r_0} \left(|f(z,y)| + \sum_{n=0}^{\infty} \left| \Delta^1 f(z, y+n) \right| \right) \sum_{k=2}^{\infty} k^\mu (1-\Delta)^k \ . \tag{3.10}$$

Now

$$\sum_{k=2}^{\infty} k^\mu (1-\theta)^k < \sum_{k=0}^{\infty} (k+1)(k+2)\cdots(k+\mu)(1-\theta)^k = \frac{d^\mu}{d\theta^\mu} \frac{1}{\theta} = (-1)^\mu \frac{\mu!}{\theta^{\mu+1}} \ .$$

Applying this to (3.10) we arrive at inequality (3.7). We prove inequality (3.8) completely analogously, using $\Delta^s u(z,y) = \sum_{k=2}^{\infty} z^k \Delta^s u_k(y)$ and corollary 3.2. Finally, using equations (3.9) and (3.6), we get that the formal power series $u_\infty(z,y) = \lim_{n\to\infty} u(z,y+n)$ constructed in theorem 3.1 has the form

$$u_\infty(z,y) = \sum_{k=2}^{\infty} \frac{f_{\infty,k}(y)}{\lambda^k - \lambda}$$

and defines an analytic function which satisfies (3.5). □

Theorem 3.4. *Let $Y = \{y: y \geq 1\}$. Suppose the following hold:*

1. $\lambda = e^{2\pi i \Delta}$ *satisfies equation (1.10);*
2. $\lim_{y\to\infty} f(z,y) = f(z,\infty)$ *exists;*
3. *for some $l \in \mathbb{N}$ and $\beta \in \mathbb{R}$ satisfying $\beta \geq 2l$ the function $f(z,x^{-\beta})$ has $l+2$ derivatives in x in the domain $|z| \leq r_0, 0 \leq x \leq 1$ and in this domain for any $s \in \mathbb{N}$ satisfying $1 \leq s \leq l+2$ the function $\frac{\partial^s}{\partial x^s} f(z,x^{-\beta})$ is analytic in z.*

Then for any θ satisfying $0 < \theta < 1$ equations (3.2) and (3.3) define a solution $u(z,y) = u(z,x^{-\beta})$ of equation (1.12) which is z-analytic, has l partial derivatives with respect to x in the domain $|z| < r_0(1-\theta), 0 \leq x \leq 1$. Furthermore in this domain the following inequalities hold:

$$|u(z,y)| \leq \frac{C_1}{\theta^{\mu+1}} \left| f(z,x^{-\beta}) \right|_1 , \tag{3.11}$$

$$\left| u(z,x^{-\beta}) \right|_s \leq \frac{C_1}{\theta^{2\mu+1}} \left| f(z,x^{-\beta}) \right|_{s+2} , \tag{3.12}$$

where $1 \leq s \leq l$, C_1 is a constant not depending on the function f and the norm $\left| f(z,x^{-\beta}) \right|_{s+2}$ is taken in the domain $|z| \leq r_0, 0 \leq x \leq 1$.

Proof. It is clear that it is enough to consider the case when the constant μ in equation (1.10) is a natural number. Now, from the assumptions of theorem 3.4 for $1 \leq s \leq l+2$ the function $\frac{\partial^s}{\partial x^s} f(z,x^{-\beta})$ which is z-analytic in the domain $|z| \leq r_0$ has

$$\frac{\partial^s}{\partial x^s} f_k(y) = \frac{\partial^s}{\partial x^s} f_k(x^{-\beta})$$

as it's z^k Taylor coefficient and Cauchy's estimate yields

$$\left|\frac{\partial^s}{\partial x^s} f_k(x^{-\beta})\right| \leq \frac{|f|_s}{r_0^k} \ . \tag{3.13}$$

Let $\alpha = \beta^{-1}$. In the disc $|z| < r_0(1-\theta)$ for $x \neq 0$ equations (3.2, 3.3, 3.9, 3.13) yield

$$\begin{aligned} |u(z,y)| &\leq \frac{C_0}{4}\left(|f|_0 + \alpha|f|_1 \sum_{n=0}^{\infty} \frac{1}{(y+n)^{1+\alpha}}\right) \sum_{k=2}^{\infty} k^{\mu}(1-\theta)^k \\ &< \frac{C_0 \mu!}{2\theta^{\mu+1}} \left|f(z, x^{-\beta})\right|_1 \ . \end{aligned}$$

This proves inequality (3.11) for $x \neq 0$. Also

$$\left|\sum_{n=0}^{\infty} \Delta^1 f_k(y+n)\lambda^{(k-1)(n+1)}\right| < \frac{\alpha|f|_1}{r_0^k} \sum_{n=0}^{\infty} \frac{1}{(y+n)^{1+\alpha}} \to 0$$

as $y \to \infty$. Thus using equation (3.3) the limit

$$\lim_{y\to\infty} u(z,y) = u(z, x^{-\beta}) \mid_{x=0} = \sum_{k=2}^{\infty} \frac{z^k}{\lambda^k - \lambda} f_k(\infty)$$

exists and satisfies the estimate (3.11).

We go on to prove inequality (3.12). For this we use the following two equations which were proven in [**5**] (§2, proof of Lemma 2.3):

$$\left|\frac{\partial^s}{\partial x^s} u(z,y)\right| \leq \sum_{k=2}^{\infty} \frac{|z|^k}{|\lambda^k - \lambda|}\left(|f_k|_s + \frac{C_2 |f_k|_{s+2}}{|\lambda^k - \lambda|}\right) \ , \tag{3.14}$$

(here C_2 is a positive constant not depending on f) and

$$\lim_{x\to 0} \frac{\partial^s}{\partial x^s} u(z, x^{-\beta}) = \sum_{k=2}^{\infty} \frac{z^k}{|\lambda^k - \lambda|} \frac{\partial^s}{\partial x^s} f_k(x^{-\beta}) \ . \tag{3.15}$$

Now using equations (3.13, 3.9, 3.14) in the disc $|z| < r_0(1-\theta)$ we have the inequality:

$$\left|\frac{\partial^s}{\partial x^s} u(z,y)\right| \leq \frac{C_0 + C_0^2 C_2}{4} |f|_{s+2} \sum_{k=2}^{\infty} k^{2\mu}(1-\theta)^k \ . \tag{3.16}$$

Plugging in

$$\left|\sum_{k=2}^{\infty} k^{2\mu}(1-\theta)^k\right| < \left|\frac{d^{2\mu}}{d\theta^{2\mu}} \sum_{k=0}^{\infty} (1-\theta)^k\right| = \frac{(2\mu)!}{\theta^{2\mu+1}}$$

into equation (3.16) yields inequality (3.12) for $x \neq 0$. Finally, since the function $u(z, x^{-\beta})$ is continuous for $x = 0$, from equation (3.15) we get that it is s times differentiable in x at $x = 0$ and equation (3.12) is fulfilled at $x = 0$. Theorem 3.4 is proven. □

4. Linearizability of NDS with discrete time

Our first theorem states that a NDS given by equation (1.1) can be linearized using a change of variables given by a formal power series in z.

Theorem 4.1. *Suppose λ is not a root of unity and that the function $f^{(n)}(z)$ from (1.1) satisfy the inequality (1.13). Then there exists a formal change of variables $\zeta^{(n)} = P^{(n)}(z), n = 1, 2, \ldots$ of the form described by equation (1.3) which give the formal power series (1.4). Similarly the inverse formal change of variables $z = V^{(n)}(\zeta^{(n)})$ give rise for each $n \in \mathbb{N}$ to the formal equation $\zeta^{(n+1)} = P^{(n+1)}(F^{(n)}(V^{(n)}\zeta^{(n)})))$ and formally give rise to the equation $\zeta^{(n+1)} = \lambda\zeta^{(n)}$.*

Proof. To find the formal power series $P^{(n)}(z)$ and $V^{(n)}(\zeta^{(n)})$ we must find a formal power series solution to the following system of functional equations:

$$V^{(n+1)}(\lambda\zeta) = F^{(n)}(V^{(n)}(\zeta)), \ n = 1, 2, \ldots \ . \tag{4.1}$$

Here $F^{(n)}(z)$ was introduced in equation (1.1) and both sides of equation (4.1) are formal power series in ζ. From equation (1.3) we see that the linear term in the series $P^{(n)}(z)$ are equal to z. Thus using equation (1.1) the linear term in both parts of the equality (4.1) are equal to $\lambda\zeta$. Suppose now that for any natural k

$$V^{(n)}(\zeta) = \zeta + \sum_{k=2}^{\infty} v_k^{(n)} \zeta^k \ . \tag{4.2}$$

Substituing these expressions for $V^{(n)}(\zeta)$ in equation (4.1) we will search for the coefficient $v_k^{(n)}(k = 2, 3, \ldots)$ which equates the monomials with the identical power of ζ in both sides of equation (4.1). We represent the function $f^{(n)}(z)$ which was defined by (1.1) as a power series the following way:

$$f^{(n)}(z) = \sum_{k=2}^{\infty} f_k(z,n),\, f_k(z,n) = f_k^{(n)} z^n \ , \tag{4.3}$$

where $n = 1, 2, \dots$ and $f_k^{(n)}$ do not depend on z. Then, using inequality (1.13), Cauchy estimates and the definition of the sth difference $\Delta^s h(z,y)$ we have for each $k \geq 2$

$$\sum_{k=1}^{\infty} \sup_{|z| \leq r_0} \left| \Delta^1 f_k(z,n) \right| < \infty \ . \tag{4.4}$$

Let $k = 2$. Comparing the monomial with ζ^2 in both parts of equation (4.1) gives that the function $u_2(\zeta, n) = v_2^{(n)} \zeta^2$ satisfy the following equation

$$u_2(\lambda\zeta, n+1) = \lambda u_2(\zeta, n) + f_2(\zeta, n) \ , \tag{4.5}$$

which has the form of equation (1.12). By (4.4) the function $f(\zeta, n) = f_2(\zeta, n)$ satisfies condition (3.1) for $\zeta = z, n = y$. Thus using theorem 3.1 there exists a solution $u_2(\zeta, n) = v_2^{(n)} \zeta^2$ satisfying equation (4.5) for which

$$\sum_{n=1}^{\infty} \left| v_2^{(n+1)} - v_2^{(n)} \right| < \infty \ .$$

We inductively assume that for each integer $m \geq 2$ and $n = 1, 2, \dots$ that the coefficients $v_k^{(n)}$ of the series $V^{(n)}(\zeta)$ in (4.2) ($2 \leq k \leq m$) are defined so that the monomials with ζ^k in both parts of equation (4.1) coincide and satisfy the inequality

$$\sum_{n=1}^{\infty} \left| v_k^{(n+1)} - v_k^{(n)} \right| < \infty \ . \tag{4.6}$$

We express (4.1) in the form

$$V^{(n+1)}(\lambda\zeta) = \lambda V^{(n)}(\zeta) + F^{(n)}(V^{(n)}(\zeta)) - \lambda V^{(n)}(\zeta)) \ .$$

We obtain that the function $u_{m+1}(\zeta, n) = v_{m+1}^{(n)} \zeta^{m+1}$ satisfies the equation

$$u_{m+1}(\lambda\zeta, n+1) = \lambda u_{m+1}(\zeta, n) + \hat{F}_{m+1}(\zeta, n) \tag{4.7}$$

where $\hat{F}_{m+1}(\zeta, n) = \hat{F}_{m+1}^{(n)} \zeta^{m+1}$ is a monomial, in which $\hat{F}_{m+1}^{(n)}$ is a number, independent of ζ, which is the value of a polynomial function of $f_k^{(n)}(k = 2, 3, \dots, n+1)$ and $v_k^{(n)}(k = 2, \dots, m)$. Note that the $v_k^{(n)}$ are already known by the inductive hypothesis. Using this and inequalities (4.4) and (4.6) we see that the function $f(\zeta, n) = \hat{F}_{m+1}(\zeta, n)$ (for $\zeta = z, n = y$) satisfies inequality (3.1) and by theorem 3.1 there exists a solution $u_{m+1}(\zeta, n) = v_{m+1}^{(n)}\zeta^{m+1}$ satisfying equation (4.7), for which

$$\sum_{n=1}^{\infty} \left| v_{m+1}^{(n+1)} - v_m^{(n)} \right| < \infty .$$

In such a fashion we get the change of variables $z = V^{(n)}(\zeta^{(n)}) = \zeta^{(n)} + \dots$ in the form of a formal power series beginning with the linear term $\zeta^{(n)}$. Thus the inverse change of variables $\zeta^{(n)} = P^{(n)}(z) = z + \dots$ is defined. Using equation (4.6) $P^{(n)}$ converges as $n \to \infty$ to the formal change of variables (1.4) and formally solves equation (1.6). Theorem 4.1 is proven.

Next, we will formulate and prove theorems on the analytic linearizability of nonautonomous dynamical systems of the form (1.1) (theorem 4.2) and (1.8) (theorem 4.5) and obtain corollary 4.6 to theorem 4.5 on the Lyapunov stability of the fixed point $z = 0$.

Theorem 4.2. *Suppose that Δ, from the expression $\lambda = e^{2\pi i \Delta}$ satisfies equation (1.10). Furthermore suppose that the function $F^{(n)}$ defined in (1.1) satisfies conditions i)-iv) of condition (L) for $l = 16$ and some $\beta > 0$. Then the nonautonomous dynamical system of the form (1.1) is analytically linearizable in the following sense: there exists positive constants r_1, r_2 and a sequence of changes of variables $\zeta^{(n)} = P^{(n)}(z)$ of the form (1.3) which are analytic in the disc $|z| \leq r_1$ and as $n \to \infty$ they converge to the change of variables (1.4) in the disc. For any $n \in \mathbb{N}$ the mapping $D^{(n)} = P^{(n+1)} \circ F^{(n)} \circ (P^{(n)})^{-1}$ is defined in the disc $\left|\zeta^{(n)}\right| \leq r_2$ and satisfies equation (1.6).*

The proof of theorem 4.2 is connected to condition (1.10) and theorem 3.4. The proof coincides with the proof of theorem 1 from [**5**] up to the changes we will indicate here. In the same way that the proof of theorem 1 from [**5**] follows from theorem 3, the proof of theorem 4.2 follows from the following theorem.

Theorem 4.3. *Consider the mapping* $F:(z,y) \to (z',y')$ *of the form* $z' = F(z,y) = \lambda z + f(z,y), y' = y+1$ *defined in the domain* $|z| \le r_0, y \ge 1$. *Suppose that* Δ *satisfies condition (1.10), and that the function* $F(z,y)$ *satisfies parts i)-iv) of condition (L) for* $l = 16$ *and some* $\beta \ge 32$. *Then, for some* $r > 0$ *there exists a* ζ*-analytic in the disc* $|z| \le r$ *and* y*-continuous for* $y \ge 1$ *function* $U(\zeta,y) = \zeta + u(\zeta,y)$ *for which*

1. $u(0,y) = \frac{\partial u}{\partial \zeta}(0,y) = 0$ *for* $y \ge 1$*; there exists* $u(\zeta,\infty) = \lim_{y\to\infty}(\zeta,y)$ *and the converges is uniform in the disc* $|\zeta| \le r$*;*

2. *The inverse change of variables* $z = U(\zeta,y)$ *is defined for* $y \ge 1$ *in the disc* $|\zeta| \le r$*;*

3. *The map* F *expressed in the coordinates* ζ, y *is defined in the domain* $|\zeta| \le r, y \ge 1$ *and has the form* $\zeta' = \lambda\zeta, y' = y+1$.

It is clear that under the conditions of theorem 4.3 the change of variables must satisfy the functional equation

$$U(\lambda\zeta, y+1) = F(U(\zeta,y),y) \ .$$

Similarly to the proof of theorem 3 in [**5**] we construct a sequence of newtonian changes of variables for the solution. For $l = 16$ we introduce parameters, $N_n, \epsilon_n, r_n, \theta_n (n = 1,2,\dots)$ which satisfy the following relations:

$$\begin{aligned} &N_{n+1} = N_n^{\frac{3}{2}},\ \epsilon_n = N_n^{-5},\ r_n = \frac{\hat r}{2}(1+2^{-n}),\ \theta_n = \frac{1}{12(2^n+1)}\ ; \\ &N_1^{-1} < \theta_1^{4(2\mu+1+l)}\ , \end{aligned} \tag{4.8}$$

where $\mu \in \mathbb{N}$ was introduced in equation (1.10) and $\hat r \le \frac{4}{3} r_0$.

We will use the following inductive lemma.

Lemma 4.4. *Suppose that the mappings*

$$F_n : \begin{cases} z' = F_n(z,y) = \lambda z + f_n(z,y) \\ y' = y+1 \end{cases} ,$$

where $f_n(z,y)$ *is a series in* z *starting with second order terms, is defined for* $|z| < r_n, y \ge 1$ *and satisfies*

1. *The function* $F(z,y) = F_n(z,y)$ *satisfies the conditions of theorem 4.3;*

2. $\left|\frac{\partial f_n}{\partial z}\right| < \epsilon_n$, $|f_n|_\rho < N_n^\rho r_n$ *for* $1 \le \rho \le l$.

Then there exists a constant $\bar{N}$ *so that if* $N_n \ge \bar{N}$ *then there exists a* ζ*-analytic and* y*-continuous change of variables* $z = V_n(\zeta, y) = \zeta + v_n(\zeta, y)$ *defined in the domain* $|\zeta| < r_n(1 - 2\theta_n)$ *for which the following hold:*

1. $v_n(0, y) = \frac{\partial v_n}{\partial \zeta}(0, y) = 0$, $\left|\frac{\partial v_n}{\partial \zeta}\right| < N_n^{-2}\theta_n$, $\lim_{y \to \infty} v_n(\zeta, y) = v_n(\zeta, \infty)$ *exists, and the convergence is uniform in the disc* $|\zeta| < r_n(1 - 2\theta_n)$*;*

2. *In the domain* $|\zeta| < r_{n+1}$, $1 \le y \le \infty$ *the mapping*

$$F_{n+1} = V_n^{-1} \circ F_n \circ V_n \colon \begin{cases} \zeta' = F_{n+1}(\zeta, y) = \lambda\zeta + f_{n+1}(\zeta, y) \\ y' = y + 1 \end{cases},$$

where $\zeta' = V_n^{-1}(z', y + 1)$, *is the inverse change of variables to* $z' = V_n(\zeta', y + 1)$, $f_{n+1}(0, y) = \frac{\partial f_{n+1}}{\partial \zeta}(0, y) = 0$, *and the following inequalities hold:*

$$\left|\frac{\partial f_{n+1}}{\partial \zeta}\right| < \epsilon_{n+1}, \left|f_{n+1}(\zeta, x^{-\beta})\right| < N_{n+1}^\rho r_n \quad \textit{for } 1 \le \rho \le l \ .$$

The proof of the inductive lemma uses theorem 3.4 and inequality (4.8) exactly like in section 5 of [**5**] the inductive lemma was proven using lemma 2.1 and 2.3 from section 2 of [**5**] (instead of our theorem 3.4) and the inequality $N_1^{-1} < \theta_1^{(l+5)4}$ from section 5 of [**5**] (instead of our inequality (4.8)).

Using the inductive lemma we get a sequence of changes of variables $V_1, V_2, \dots$ and a sequence of mappings $F = F_1, F_2, \dots$, which for $n \ge 1, \zeta^{(1)} = z, \zeta^{(n)} = V_n(\zeta^{(n+1)}, y) = \zeta^{(n+1)} + v_n(\zeta^{(n+1)}, y), F_{n+1} = V_n^{-1} \circ F_n \circ V_n$.

Thus, for $n \ge 1$ the change of variables $U_{n+1} = V_1 \circ V_2 \circ \cdots \circ V_n$ conjugates F to $F_n = U_n^{-1} \circ F \circ U_n$. From the inductive lemma it easily follows that if $\hat{r} > 0$ is sufficiently small, $N_1 > \bar{N}$ and the inequalities

$$\left|\frac{\partial f}{\partial z}\right| < \epsilon_1 = N_1^{-5} \ , |f|_\rho < N_1^\rho r_1 \text{ for } 1 \le \rho \le l$$

hold in the domain $|z| < r_1 = \frac{3\hat{r}}{4} < r_0$ then the mappings U_n and F_n are defined in the domain $\left|\zeta^{(n)}\right| < r_n, y \ge 1$ and the sequence of functions $U_n(\zeta, y)$ in the domain $|\zeta| < \frac{\hat{r}}{2}, y \ge 1$ uniformly converge (as $n \to \infty$)

to the function $U(\zeta, y) = \zeta + u(\zeta, y)$ for which $u(0, y) = \frac{\partial u}{\partial \zeta}(0, y) = 0$ and $\lim_{y\to\infty} u(\zeta, y) = u(\zeta, \infty)$.

Thus for sufficiently small $r > 0$ in the disc $|z| \le r$ the change of variables $z = \zeta + u(\zeta, y)$ is invertible, and by conclusion 2 of the inductive lemma: in the domain $|\zeta| \le r, y \ge 1$ we have $\lim_{n\to\infty} F_n(\zeta, y) = \lambda\zeta$, we have in this domain the change of variables $z = U(\zeta, y)$ brings F to

$$U^{-1} \circ F \circ U = \lim_{n\to\infty} F_n\colon \begin{cases} \zeta' = \lambda\zeta \\ y' = y + 1 \end{cases}.$$

This completes the proof of theorem 4.3. □

To derive theorem 4.2 from theorem 4.3 we point out, that if condition (L) holds for the sequence of mappings $F^{(n)}$ for $l = 16$ and $\beta > 0$, then it will hold for $l = 16$ and $\beta_1 = m\beta$, where m is an arbitrary natural number. Thus, if the conditions of theorem 4.2 hold for $l = 16, \beta > 0$, then the conditions of theorem 4.3 applied to the mapping

$$F\colon \begin{cases} z' = F(z, y) = \lambda z + f(z, y) \\ y' = y + 1 \end{cases}$$

will hold for $l = 16$ and $\beta_1 = m\beta \ge 32$ for some m. Now theorem 4.2 follows directly from theorem 4.3 with $y = 1$. □

Theorem 4.5. *Suppose that Δ defined by $\lambda = e^{2\pi i\Delta}$ satisfies condition (1.10) and the function $F^{(n)}$ defined in (1.8) satisfies parts i)-vii) of condition (L) with $l = 16$ and some $\beta > 0$. Then the nonautonomous dynamical system defined by equations (1.8) is linearizable with a z-analytic change of coordinates depending on time n in the following sense. For some positive constants r_1, r_2, there exists an sequence of changes of coordinates $\zeta^{(n)} = P^{(n)}(z)$ of the form (1.9), which are analytic and invertible in the disc $|z| \le r_1$. As $n \to \infty$ in this disc they converge to the change of variables (1.4) and for any $n \in \mathbb{N}$ the mapping $D^{(n)} = P^{(n+1)} \circ F^{(n)} \circ (P^{(n)})^{-1}$ is defined in the disc $\left|\zeta^{(n)}\right| \le r_2$. Finally equation (1.6) holds.*

Proof. The proof of theorem 4.5 is based on the fact that using a linear change of variables $\zeta = \delta(y)z$ the mapping

$$F\colon \begin{cases} z' = F(z, y) = \lambda(y)z + f(z, y) \\ y' = y + 1 \end{cases}$$

reduces to the form

$$\zeta' = G(\zeta, y) = \lambda\zeta + g(\zeta, y), y' = y + 1 \ , \tag{4.9}$$

where $g(0,y) = \frac{\partial g}{\partial \zeta}(0,y) = 0$ and λ does not depend on y and was introduced in part v) of condition (L). It is easy to see that for this we can take $\delta(y) = \Lambda(y)$ where $\Lambda(y)$ was introduced in part vi) of condition (L). Applying parts v)-vii) of condition (L), we get that the sequence of mappings arising from the mapping (4.9) with $y = n, (n = 1, 2, \dots)$ satisfy the conditions of theorem 4.2, and thus theorem 4.5 follows. □

Corollary 4.6. *If for the nonautonomous dynamical system of the form (1.8) with the parameter $\lambda = e^{2\pi i\Delta}$ satisfying condition (1.10) condition (L) holds for $l = 16$ and some $\beta > 0$ then the fixed point $z = 0$ is stable in the sense of Lyapunov.*

Proof. Let $z = (P^{(1)})^{-1}(\zeta^{(1)})$, $F^{(n)} \circ F^{(n-1)} \circ \cdots \circ F^{(1)} z = z^{(n)}$, $\zeta^{(n+1)} = P^{(n+1)}(z^{(n)})$, $\left|\zeta^{(1)}\right| \leq r_2$, where the change of variables $P^{(n)}$ and the constant r_2 where introduced in theorem 4.5. Then, using theorem 4.5 for any $n \geq 1$ we have

$$\left|\zeta^{(n+1)}\right| = \left|P^{(n+1)} z^{(n)}\right| = \left|\prod_{k=1}^{n} \lambda^{(k)}\right| \left|\zeta^{(1)}\right| , \tag{4.10}$$

and using vi) of condition (L) we have

$$\left|\prod_{k=1}^{n} \lambda^{(k)}\right| < \hat{c} \ , \tag{4.11}$$

where $\hat{c}$ is a constant independent of n. Thus the corollary follows from (1.9, 4.10, 4.11). □

Corollary 4.7. *Under the conditions of corollary 4.6 the fixed point $z = 0$ is uniformly stable in the sense of Lyapunov.*

The proof of the corollary is the same as the proof of corollary 4.6.

5. The rational case

In this section no assumption is made on the smoothness of the convergence. Our theorems are in the case that the NDS is of the more general form (1.8).

Theorem 5.1. *If the limit function F is not linear for the NDS (1.8) and λ is a root of unity then in the neighborhood of the fixed point $z = 0$ either*

(i) *there exists $z_0 \neq 0$ such that $\liminf_{n\to\infty} F^{(n)} \circ \cdots F^{(1)}(z_0) = 0$,*

(ii) *there exist positive constants ϵ, r such that for all points z in the ball $U(0,\epsilon)$ except $z = 0$, the orbit leaves the ball $U(0,r)$, i.e.*

$$\liminf_{n\to\infty} \left\| F^{(n)} \circ \cdots F^{(1)}(z) \right\| \geq r \ .$$

Corollary 5.2. *Under the assumption of theorem 5.1 the NDS (1.8) is not conjugate to a rotation.*

Option (ii) of theorem 5.1 occurs even for the simple linear example $F^{(n)}(z) := (1 + 1/n)z$. This example shows the difference between a NDS and a diffeomorphism: for a NDS a neutral fixed point (defined by $|Df|_0 = 1$) can display repelling behavior.

Proof of theorem 5.1. Let $F^q(z) = z + f_q(z)$ where $f_q(z) = \sum_{j=r}^{\infty} b_j z^j$ and $b_r \neq 0$ for some $r \geq 2$. Note that F^q is the qth iterate of the limit map F and is not the same as $F^{(q)}$. In fact, it is known that $r \geq q+1$[2]. We make a change of coordinates which we call the inverse coordinates:

$$w = \frac{1}{\bar{z}^{r-1}} \ . \tag{5.1}$$

Let $G(w)$ (resp. $G^{(n)}(w)$) be the mapping in inverse coordinates corresponding to $F(z)$ (resp. $F^{(n)}(z)$). In the inverse coordinates the transformation $F^q(z)$ becomes

$$\begin{aligned} G^q(w) &= w - (r-1)\bar{b}_r - (r-1)\bar{b}_{r+1}w^{-1/(r-1)} + O(w^{-2/(r-1)}) \\ &:= w + c_r + c_{r+1}w^{-1/(r-1)} + O(w^{-2/(r-1)}) \end{aligned} \tag{5.2}$$

where $c_r \neq 0$. For $n \in \mathbb{Z}^+ = \{0, 1, \dots\}$ let

$$B^{(n)} := B_q^{(n)} := F^{(q(n+1))} \circ \cdots \circ F^{(qn+1)} \ .$$

By continuity we have

$$B^{(n)}(z) = \prod_{i=qn+1}^{q(n+1)} \lambda_i z + \sum_{j=2}^{\infty} b_j^{(n)} z^j \tag{5.3}$$

where $b_j^{(n)} \to b_j$ as $k \to \infty$ for all j and $b_2, \ldots, b_{r-1}$ are understood to equal 0. In other words $B^{(n)}(z)$ converges uniformly to $F^q(z)$ in a fixed neighborhood of the point $z = 0$.

For $n \in \mathbb{Z}^+$ let $C^{(n)} := C_q^{(n)} := G^{(q(n+1))} \circ \cdots \circ G^{(qn+1)}$. Now

$$C^{(n)} w = \sum_{i=-1}^{r-1} c_{r-i}^{(n)} w^{i/(r-1)} + O(w^{-2/(r-1)}) \ . \tag{5.4}$$

Here $c_1^{(n)} \to 1$, $c_k^{(n)} \to 0$ for $k = 2, \ldots r-1$, $c_r^{(n)} \to c_r$ and $c_{r+1}^{(n)} \to c_{r+1}$ and thus we can say that equation (5.4) converges to (5.2). Here we can assume that the O term does not depend on n. Of course we could express the coefficients $c_r^{(n)}$ in terms of the coefficients $f_k^{(m)}$ for $m \in \{qn+1, \ldots, q(n+1)\}$, but this is not necessary for our purposes.

Let $\mathcal{H}^+(c_r)$ be the half plane which contains the point $c_r \in \mathbb{C}$ defined by the line going through the origin orthogonal to c_r (thought of as a vector) and $\mathcal{H}^-(c_r)$ the other half plane.

Now since equation (5.4) converges to (5.2) in the limit we immediately see that there exists a $K > 0$ such that for each w satisfying $\|w\| > K$ there is a $N = N(w)$ such that for $n \geq N$

$$\left\|C^{(n)} w\right\| \geq \|w\| + \frac{\|c_r\|}{3} \ . \tag{5.5}$$

Let Ω be the domain of definition of G. For $w_0 \in \Omega$ we consider the sequence $w_n := G^{(n)} \circ \cdots \circ G^{(1)} w_0$ and the corresponding sequence $z_n := F^{(n)} \circ \cdots \circ F^{(1)} z_0$.

We assume that case (ii) of the theorem does not hold, namely that

$$\sup_{w_0 \in \Omega} \limsup_{k \to \infty} \|w_k\| = \infty \ . \tag{5.6}$$

We need to show that there is a point $w_0 \in \Omega$ for which the orbit goes to infinity, that is $\limsup_{k\to\infty} \|w_k\| = \infty$. We assume the opposite, namely for each point $w_0 \in \Omega$ the orbit stays bounded, or $\limsup_{k\to\infty} \|w_k\| = K(w_0) < \infty$. Using equation (5.6) we can fix $w_0 \in \Omega$ with $K(w_0)$ sufficiently large. Then for each $\epsilon > 0$ there is a point $v \in \mathbb{C}$ with $\|v\| = K(w_0)$ such that $w_{n_i} \in U(v, \epsilon)$ for infinitely many n_i.

For sufficiently large k by equation (5.5) we have

$$\left\|C^{(n)} w_{n_i}\right\| \geq \left\|w_{n_i}\right\| + \|c_r\|/3.$$

Now we apply (5.5) in two cases, first assume that $v \in \mathcal{H}^+(c_r)$. Now if n_i is sufficiently large by equation (5.5) we get

$$\left\|w_{n_i+q}\right\| \geq \left\|w_{n_i}\right\| + \|c_r\|/3 \tag{5.7}$$

and the right had side is large than $K(w_0)$ if ϵ is sufficiently small. This contradicts the definition of $K(w_0)$. The proof in the case $v \in \mathcal{H}^-(c_r)$ is similar, we just replace w_{n_i+q} by w_{n_i-q} in equation (5.7). □

Theorem 5.3. *Under the conditions of theorem 5.1 there exists a constant $R > 0$ such that for every $\epsilon > 0$ satisfying $\epsilon < R$ there exists a positive integers $n_i = n_i(\epsilon)$ $(i = 0, 1)$ and a point z with $|z| = \epsilon$ such that $\left|F^{(n_1)} \circ F^{(n_1-1)} \circ \cdots \circ F^{(n_0)}(z)\right| \geq R$.*

Corollary 5.4. *Under the conditions of theorem 5.1 the NDS (1.8) is not uniformly stable in the sense of Lyapunov.*

Proof of theorem 5.3. We use the notation of the proof of theorem 5.1. We assume without loss of generality that $c_r = 1$. The limit map F^q is unstable in the sense of Lyapunov [**2**]. Thus, there exists $R > 0$ such that for every $\epsilon > 0$ there is a positive integer N and a point z_0 with $|z_0| = \epsilon$ such that $\left|F^{qN} z_0\right| > R$. Let $w_0 = \bar{z}^{1-r}$, then we have

$$\left|G^{qN}(w_0)\right| < R^{1-r} . \tag{5.8}$$

Set $n_1 = n_0 + N$. We can choose n_0 so large that

$$\left|C^{(n_1)} \circ \cdots \circ C^{(n_0)}(w_0) - G^{qN}(w_0)\right| < R^{1-r}/2 . \tag{5.9}$$

Using equations (5.8) and (5.9) we immediately have

$$\left|C^{(n_1)} \circ \cdots \circ C^{(n_0)}(w_0)\right| < R^{1-r}/2 .$$

This inequality, when rewritten is z coordinates finishes the proof. □

Now we turn back to the case that the NDS of the form (1.1). In a special case we can prove more:

Theorem 5.5. *Assume* $\lambda = 1$ *and* $f_2^{(n)} \neq 0 \neq f_2$ *for all* $n \in \mathbb{N}$. *The following hold in the neighborhood of the fixed point* $z = 0$

(i) *there exists* $z_0 \neq 0$ *such that* $\lim_{n\to\infty} F^{(n)} \circ \cdots \circ F^{(1)}(z_0) = 0$, *and*

(ii) *there exist positive constant* r *such that for every* $\epsilon > 0$ *there is a point* z_0 *in the ball* $U(0, \epsilon)$ *and a* $n > 0$ *such that*

$$\left\|F^{(n)} \circ \cdots \circ F^{(1)}(z_0)\right\| \geq r \ .$$

Remark. The speed of convergence (expressed in inverse coordinates) of the orbit of z_0 to 0 in part (i) is linear as can be seen in formula (5.12).

Corollary 5.6. *Under the assumptions of theorem 5.5 the NDS (1.1) is not stable in the sense of Lyapunov.*

Proof of theorem 5.5. We use the notation of the proof of theorem 5.1. For part (i) we note that in this case the mapping $C^{(n)}$ given by equation (5.4) becomes

$$C^{(n)}w = w + c_r^{(n)} + c_{r+1}^{(n)} w^{-1/(r-1)} + O(w^{-2/(r-1)}) \tag{5.10}$$

where $c_r^{(n)} \to c_r \neq 0$ and $c_{r+1}^{(n)} \to c_{r+1}$. The O term can be assumed to be uniform in n. We can assume without loss of generality that $c_r = 1$ and rewrite this as

$$C^{(n)}w = w + 1 + \delta^{(n)} + c_{r+1}^{(n)} w^{-1/(r-1)} + O(w^{-2/(r-1)}) \ . \tag{5.11}$$

We will show that for some $w_0 \in \Omega$ and for all $n \in \mathbb{N}$

$$\|w_n - w_0 - n\| \leq o(n) \ . \tag{5.12}$$

We first assume that the NDS has the special form $C^{(n)}w = w + 1 + \delta^{(n)}$. In this case, since the $\delta^{(k)}$ converges to 0 we have $\sum_{k=1}^n \delta^{(k)} = o(n)$ and equation (5.12) follows in this case. Now suppose the NDS has the form $C^{(n)}w = w + 1 + \delta^{(n)} + c_{r+1}^{(n)} w^{-1/(r-1)}$. Set

$$C = \sup_n \left| c_{r+1}^{(n)} - c_{r+1} \right| ,$$

clearly C is finite. Then the term $c_{r+1}^{(n)} w^{-1/(r-1)}$ contributes in the first n steps at most $C \sum_{k=1}^n 1/(w + k + o(k))$. But there is a $N > 0$ such that for $k \geq N$ we have $|o(k)| \leq k/2$. Thus the contribution of this term is bounded by

$$C \sum_{k=1}^{N-1} 1/(w + k + o(k)) + \sum_{k=N}^{n} 1/(w + k/2)^{1/(r-1)}$$

and this is of the order $o(n)$. In the general case the term $O(w^{-2/(r-1)})$ contributes a lower order error term and so we have shown formula (5.12) whenever the infinite orbit w_n is defined and stays outside a sufficiently large disk $U(0, R)$. By continuity the region $\Omega_n := G^{(n)} \circ \cdots \circ G^{(1)}\Omega$ is nomempty and contains a neighborhood of the point infinity for all $n \in \mathbb{N}$. Thus part (i) follows easily from (5.12).

Now we turn to part (ii). Let $R = 1/r$ for r sufficiently small to have $\Omega \subset \mathbb{C} \backslash U(0, R)$. Consider the line $I_C := \{w: Re(w) = C\}$. In the proof of part (i) we showed that for any C, for sufficiently large positive values of $Im(w)$ the orbit w_n goes to infinity, in fact $Re(w_n) \to \infty$. Completely analogously, for sufficiently large negative values of $Im(w)$ one can show that $Re(w_n)$ also goes to infinity. Let S_R be the half strip bounded by $Im(w) = \pm R$ with $Re(w) < 0$. The proof of the above paragraph shows that w_n never enters the set $S_R \cup U(0, R)$ for $w_0 \in I_C$ for $|Im(w_0)|$ sufficiently large.

Now we argue that for any $C < -R$ by continuity that the orbit of some initial point $w \in I_C$ must hit the ball $U(0, R)$. Consider the orbit $I_n := G^{(n)} \circ \cdots G^{(1)} I_C$ of the line I_C. Note that I_n is always a Jordan curve. Let $Y := \liminf_{n \to \infty} \inf_{z \in I_n} Re(z)$. We must show that $Y \geq -R$. But if not, then we have a direct contradiction since equation (5.11) implies that $Y \geq Y + 1/2$. □

6. Behavior at infinity

Consider a sequence of conformal mappings

$$G^{(1)}, G^{(2)}, \ldots, G^{(n)}, \ldots \tag{6.1}$$

defined on the domain $Re w \geq \kappa_0$, and having the form:

$$G^{(n)}: w \to w' = w + 1 + \frac{b_n}{w} + g^{(n)}(w); \ (n = 1, 2, \ldots) \ , \tag{6.2}$$

where b_n is a constant such that the limit $\lim_{n\to\infty} b_n$ exists and is non infinite. Furthermore suppose that

$$\left|g^{(n)}(w)\right| < \frac{c}{|w|^2} , \tag{6.3}$$

where c is a constant not depending on n.

In this section we will formulate and prove theorem 6.1 from which follows that the nonautonomous dynamical system (6.1) brought to the form

$$\Phi^{(n+1)}(G^{(n)}(w)) = \Phi^{(n)}(w) + 1 \ ; (n = 1, 2, \dots) \tag{6.4}$$

using a w-analytic change of variables

$$\Phi^{(n)}(w) = \zeta^{(n)} \ , (n = 1, 2, \dots) \ .$$

This means that in the coordinate $\zeta^{(n)}$ the system (6.1) is a translation by 1. In particular, when the $G^{(n)}$ are all identical, the $\Phi^{(n)}$ are also identical and we recover the well known classical theorem [**2**].

Theorem 6.1. *Suppose that*

$$\sum_{n=1}^{\infty} |b_{n+1} - b_n| \log n < \infty \ . \tag{6.5}$$

Then there exist constants κ, q and a sequence of analytic functions $\Phi^{(n)}(w), (n = 1, 2, \dots)$ which for all $n \geq 1$ satisfy:

1. *The functions $\Phi^{(n)}(w)$ is defined in the domain*

$$\Omega = \{w \colon Re w \geq \kappa\} \text{ and in this domain } \left|\Phi^{(n)}(w) - w\right| < q \ , \tag{6.6}$$

where q is a constant not depending on n;

2. *$G^{(n)}(\Omega) \subset \Omega$ and (6.4) holds.*

Proof. For any $n \in \mathbb{N}$ using inequality (6.5) the following sum converges

$$\Delta_n = \sum_{s=1}^{\infty} \frac{b_{n+s-1} - b_{n+s}}{s} \ . \tag{6.7}$$

We will need the following lemma.

Lemma 6.2. *The sum $\Delta = \sum_{n=1}^{\infty} \Delta_n$ converges.*

Proof. From the definition of Δ_n (6.7) we have

$$\begin{aligned}\sum_{n=1}^{\infty} \Delta_n &= \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} \frac{b_{n+s-1} - b_{n+s}}{s} \\ &= \sum_{n=1}^{\infty} (b_n - b_{n+1}) \sum_{s=1}^{\infty} \frac{1}{s} \qquad (6.8) \\ &= \sum_{n=1}^{\infty} (b_n - b_{n+1})(\log n + O(1)) < \infty \ ,\end{aligned}$$

since by (6.5) the series in (6.8) converges absolutely. □

For $n, k \in \mathbb{N}$ we define

$$\sigma(k,n) = \sum_{s=1}^{k} \frac{b_{n+s-1}}{s}, \qquad (6.9)$$

$$d_n = \sum_{s=n}^{\infty} \Delta_s \qquad (6.10)$$

and

$$G_k^{(n)}(w) = G^{(n+k-1)} \circ G^{(n+k-2)} \circ \cdots \circ G^{(n+1)} \circ G^{(n)}(w). \qquad (6.11)$$

From lemma 6.2 and equation (6.10) it is clear that the sum defining d_n (6.10) converges and

$$d_{n+1} - d_n = -\Delta_n \ . \qquad (6.12)$$

Furthermore, from equations (6.11, 6.2, 6.3) it follows that there exists a constant κ such that if $Re w \geq \kappa$ then for all $n,\ k \in \mathbb{N}$

$$Re G_k^{(n)}(w) \geq \kappa, \ \frac{k}{2} \leq \left| G_k^{(n)}(w) \right| \leq |w| + 2k \ . \qquad (6.13)$$

From equations (6.9, 6.5) it follows that

$$|\sigma(k,n)| \leq c_1 \log k \qquad (6.14)$$

for all $n, k \in \mathbb{N}$ where c_1 is a constant, not depending on n and k.

We now want to find the change of variables $\Phi^{(n)}$ required by theorem 6.1 in the form of a limit $\Phi^{(n)}(w) = \lim_{k \to \infty} \Phi_k^{(n)}(w)$. We define the $\Phi_k^{(n)}(w)$ by:

$$\Phi_k^{(n)}(w) = G_k^{(n)}(w) - k - \sigma(k,n) + d_n \ . \qquad (6.15)$$

First of all let us prove that if κ is sufficiently large, then for all $n \in \mathbb{N}$ the sequence $\Phi_k^{(n)}(w)$ converges as $k \to \infty$ to the analytic function $\Phi^{(n)}(w)$ in the domain $Re w \geq \kappa$. Using equations (6.11, 6.2), for $k > 2$ we have:

$$G_{k+1}^{(n)}(w) - G_k^{(n)}(w) = 1 + \frac{b_{n+k}}{G_k^{(n)}(w)} + g^{(n+k)}(G_k^{(n)}(w)) - g^{(n+k-1)}(G_{k-1}^{(n)}(w)) .$$

From thus, using equations (6.15, 6.9, 6.3, 6.5, 6.13) we get:

$$\begin{aligned} \Phi_{k+1}^{(n)}(w) - \Phi_k^{(n)}(w) &= G_{k+1}^{(n)}(w) - G_k^{(n)}(w) - 1 - \frac{b_{n+k}}{k+1} \\ &= -\frac{b_{n+k}}{k+1} + \frac{b_{n+k}}{G_k^{(n)}(w)} + O\left(\frac{1}{k^2}\right) = O\left(\frac{1}{k}\right) . \end{aligned} \tag{6.16}$$

Furthermore from (6.16) we have

$$\left|\Phi_k^{(n)}(w) - w\right| \leq \left|\Phi_1^{(n)}(w) - w\right| + \sum_{s=1}^{k-1} \left|\Phi_{s+1}^{(n)}(w) - \Phi_s^{(n)}(w)\right| = O(\log k) .$$

Placing this in equation (6.16) and using equations (6.15, 6.10, 6.14) and lemma 6.2 yields:

$$\begin{aligned} \Phi_{k+1}^{(n)}(w) - \Phi_k^{(n)}(w) &= b_{n+k}\left(\frac{1}{k + \sigma(k,n) - d_n + \Phi_k^{(n)}(w)} - \frac{1}{k+1}\right) + O\left(\frac{1}{k^2}\right) \\ &= \frac{b_{n+k}}{k^2} O\left(\left|\Phi_k^{(n)}(w)\right| + |\sigma(k,n)| + |d_n|\right) = O\left(\frac{\log k}{k^2}\right) . \end{aligned}$$

Thus

$$\sum_{k=1}^{\infty} \left|\Phi_{k+1}^{(n)}(w) - \Phi_k^{(n)}(w)\right| < \infty,$$

and the sequence $\Phi_k^{(n)}(w)(k = 1, 2, \dots)$ converges to an analytic function $\Phi^{(n)}(w)$. Furthermore inequality (6.6) holds for some constant q.

From the definition of $G^{(n)}$ in equation (6.2) it clearly follows that there is a sufficiently large constant κ such that $G^{(n)}(\Omega) \subset \Omega$ where Ω was introduced in theorem 6.1. Now we will demonstrate equation (6.4). From the definitions of $\Phi_k^{(n)}(w)$ (6.15) and $G_k^{(n)}(w)$ (6.11) we have:

$$\begin{aligned} &\Phi_k^{(n+1)}(G^{(n)}(w)) = \\ &= \Phi_{k+1}^{(n)}(w) + 1 + (\sigma(k+1,n) - \sigma(k,n+1)) + d_{n+1} - d_n , \end{aligned} \tag{6.17}$$

for all $n, k \in \mathbb{N}$. Using the definition of $\sigma(k,n)$ (6.9) we have

$$\sigma(k+1,n) - \sigma(k,n+1) = \frac{b_{n+k}}{k+1} + \sum_{s=1}^{k} \frac{b_{n+s-1} - b_{n+s}}{s} \ . \tag{6.18}$$

Taking the limit as $k \to \infty$ in equation (6.17) and using equations (6.18, 6.7) yields:

$$\Phi^{(n+1)}(G^{(n)}(w)) = \Phi^{(n)}(w) + 1 + \Delta_n + d_{n+1} - d_n \ . \tag{6.19}$$

Now equations (6.4) clearly follows from equations (6.19, 6.12). □

Remark. Using the functional equation (6.4) for any $n \in \mathbb{N}$ we can analytically continue the function $\Phi^{(n)}(w)$ to any domain Ω for which the functions $G^{(n)}(w)$ are defined, $G^{(n)}(\Omega) \subset \Omega$ and $\lim_{k\to\infty} ReG_k^{(n)}(w) = \infty$ for all $w \in \Omega$.

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